Financial Econometrics A
February 17, 2020
SOLUTION KEY

Please note there is a total of **10** questions that you should provide answers to. That is, **5** questions under *Question A*, and **5** under *Question B*.

Question A:

Consider the model for $x_t \in \mathbb{R}$ (with $t = 1, 2, ..., T$) given by

$$x_t = \mu + \varepsilon_t, \quad \varepsilon_t = \sigma_t z_t,$$

with z_t i.i.d.N$(0,1)$, $x_0 = 0$ and

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2.$$

The parameters satisfy $\mu \in \mathbb{R}$, $\omega > 0$ and $\alpha \geq 0$.

Question A.1: Derive a condition under which x_t is weakly mixing with $Ex_t^2 < \infty$.

Solution: x_t is a Markov chain, with strictly positive and continuous transition density, and we use $\delta(x) = 1 + x^2$:

$$\begin{aligned} E\left(\delta\left(x_t\right)|x_{t-1} = x\right) &= 1 + E\left(\left(\mu + \varepsilon_t\right)^2 |x_{t-1} = x\right) \\ &= 1 + \mu^2 + \left(1 + \alpha x^2\right) E z_t^2 \end{aligned}$$

and we see that $\alpha < 1$ is sufficient.

Question A.2: With $\theta = (\mu, \omega, \alpha)'$ the likelihood function is given by

$$L(\theta) = -\frac{1}{2T} \sum_{t=1}^{T} \left(\log \sigma_t^2(\theta) + \frac{(x_t - \mu)^2}{\sigma_t^2(\theta)} \right),$$

with $\sigma_t^2(\theta) = \omega + \alpha x_{t-1}^2$. Show that if $\alpha_0 < 1$, then with $\theta_0 = (\mu_0, \omega_0, \alpha_0)'$ the true parameter value,

$$\sqrt{T} \partial L(\theta_0) / \partial \mu \xrightarrow{D} N(0, \xi) \quad \xi = E\left(\frac{x_t - \mu_0}{\omega_0 + \alpha_0 x_{t-1}^2}\right)^2$$

Solution:

$$\sqrt{T}\partial L\left(\theta_0\right)/\partial\mu = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{x_t-\mu_0}{\omega_0+\alpha_0 x_{t-1}^2}$$

and, using that $E[(x_t-\mu_0)/(\omega_0+\alpha_0 x_{t-1}^2)|x_{t-1}]=0$, we can apply the CLT for weakly mixing processes if

$$E\left(\frac{x_t-\mu_0}{\omega_0+\alpha_0 x_{t-1}^2}\right)^2<\infty.$$

As

$$E\left(\frac{x_t-\mu_0}{\omega_0+\alpha_0 x_{t-1}^2}\right)^2<E\left(x_t-\mu_0\right)^2/\omega_0$$

this holds if $Ex_t^2<\infty$ or $\alpha_0<1$.

Question A.3: Show that if x_t is weakly mixing with $\alpha_0>0$, then with $\theta_0=(\mu_0,\omega_0,\alpha_0)'$ the true parameter value,

$$\sqrt{T}\partial L\left(\theta_0\right)/\partial\alpha\xrightarrow{D}N\left(0,\beta\right)\quad\beta=\frac{1}{2}E\left(\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\right)^2$$

Solution:

$$\sqrt{T}\partial L\left(\theta_0\right)/\partial\alpha=\frac{1}{2\sqrt{T}}\sum_{t=1}^{T}\left(z_t^2-1\right)\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}$$

and we can apply the CLT for weakly mixing processes if

$$E\left(\left(z_t^2-1\right)\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\right)^2<\infty.$$

As $E\left(z_t^2-1\right)^2=2$, and

$$E\left(\frac{x_{t-1}^2}{\omega_0+\alpha_0 x_{t-1}^2}\right)^2<1/\alpha_0^2$$

the result holds if $\alpha_0>0$.

Question A.4: We may conclude that if $0<\alpha_0<1$ then asymptotic normality holds for $\hat{\theta}$. Argue that the limiting distribution of the LR statistic for the hypothesis that $\mu=0$ is χ^2.

Solution: This holds by the asymptotic normality of $\hat{\theta}$ using standard expansions of the LR statistic.

Question A.5: Now consider testing the hypothesis that $\alpha = 0$. In this case the asymptotic distribution of the LR statistic is "$\frac{1}{2}\chi^2$". Explain why - and explain how this is related to Questions A.2 and A.3.

Solution: As $\alpha = 0$ is a boundary point the standard theory breaks down - and the "$\frac{1}{2}\chi^2$" holds since there are no boundary issues for ω and μ. Moreover, in Question A.3 the asymptotic normality argument breaks down as $\alpha_0 > 0$ does not hold - and we see that $Ex_t^4 < \infty$ is needed (which always holds, since x_t is Gaussian if $\alpha_0 = 0$).

Question B:

Suppose that the logarithm of the price of a share of stock is given by

$$p(t) = p(0) + \mu t + \sigma W(t), \quad t \in [0, T], \tag{B.1}$$

where $p(0) \in \mathbb{R}$ is some fixed initial value, $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants, and $W(t)$ is a Brownian motion.

Recall here that the Brownian motion $W(t)$ has the properties

1. $W(0) = 0$.
2. W has independent increments, i.e. if $0 \leq r < s \leq t < u$, then

   $$W(u) - W(t) \text{ and } W(s) - W(r)$$

   are independent.
3. The increments are normally distributed, i.e.

   $$W(t) - W(s) \sim N(0, t - s)$$

   for all $0 \leq s \leq t$.

Suppose that we have observed the price $p(t)$ at $n + 1$ equidistant points

$$0 = t_0 < t_1 < \ldots < t_n = T,$$

with

$$t_i = \frac{i}{n} T, \quad i = 0, ..., n.$$

Based on these points we obtain n log-returns given by

$$r(t_i) = p(t_i) - p(t_{i-1}), \quad i = 1, ..., n.$$

Question B.1: Argue that $r(t_i)$ is normally distributed, i.e. show that

$$r(t_i) \sim N\left(\mu \frac{T}{n}, \sigma^2 \frac{T}{n}\right).$$

Show that

$$\text{cov}(r(t_i), r(t_{i-1})) = 0.$$

Solution: *The properties follow directly from the definition of* $r(t_i)$ *and the properties of the Brownian motion. Derivations should be included.*

Question B.2: We now seek to estimate the model parameters (μ, σ^2) based on maximum likelihood. Given the n log-returns, the log-likelihood function is (up to a constant and a scaling factor)

$$L_n(\mu, \sigma^2) = \sum_{i=1}^{n} \left\{ -\log(\sigma^2 \frac{T}{n}) - \frac{\left[r(t_i) - \mu \frac{T}{n}\right]^2}{\sigma^2 \frac{T}{n}} \right\}.$$

Let $\hat{\mu}$ denote the maximum likelihood estimator of μ.
Show that

$$\hat{\mu} = \frac{1}{T} \sum_{i=1}^{n} r(t_i) = \frac{1}{T} \left[p(T) - p(0)\right].$$

Argue that the sampling frequency of the log-returns over the interval $[0, T]$ does not have any influence on the estimate of μ.

Solution: *By solving the F.O.C. for maximization of* $L_n(\mu, \sigma^2)$*, that is solving*

$$\frac{\partial L_n(\mu, \sigma^2)}{\partial \mu} = 0$$

for μ*, yields the MLE*

$$\hat{\mu} = \frac{1}{T} \sum_{i=1}^{n} r(t_i).$$

Derivations should be included. Moreover,

$$\sum_{i=1}^{n} r(t_i) = \sum_{i=1}^{n} p(t_i) - p(t_{i-1}) = p(t_n) - p(t_0) = p(T) - p(0),$$

by the definition of t_i*. Hence, the MLE does not depend on* n*, i.e. the number of observations within the interval* $[0, T]$*.*

Question B.3: Assume now that $T = 1$, such that we have n observations of the log-returns over the time interval $[0, 1]$, which you may think of as the time interval over one trading day. Then the maximum likelihood estimator for σ^2 is given by

$$\hat{\sigma}^2 = \sum_{i=1}^{n} \left[r(t_i) - \frac{1}{n} \sum_{i=1}^{n} r(t_i) \right]^2.$$

Use that $r(t_i) = \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}} z_i$, with $z_i \sim i.i.d.N(0,1)$ in order to show that

$$\frac{1}{n}\sum_{i=1}^{n} r(t_i) \xrightarrow{p} 0 \quad \text{as } n \to \infty.$$

Explain briefly how $\hat{\sigma}^2$ is related to the Realized Volatility.

Solution: *We have, that*

$$\frac{1}{n}\sum_{i=1}^{n} r(t_i) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mu}{n} + \frac{\sigma}{\sqrt{n}} z_i\right)$$
$$= \frac{\mu}{n} + \frac{\sigma}{\sqrt{n}}\frac{1}{n}\sum_{i=1}^{n} z_i.$$

For the first term, $\frac{\mu}{n} \to 0$ as $n \to \infty$. For the second term, $\frac{1}{n}\sum_{i=1}^{n} z_i \xrightarrow{p} E(z_i) = 0$ by the LLN for i.i.d. processes. We conclude that $\frac{1}{n}\sum_{i=1}^{n} r(t_i) \xrightarrow{p} 0$ as $n \to \infty$.

The realized volatility (over the interval $[0,1]$) is

$$\sum_{i=1}^{n} [r(t_i)]^2 .$$

Hence the realized volatility is obtained from $\hat{\sigma}^2 = \sum_{i=1}^{n}\left[r(t_i) - \frac{1}{n}\sum_{i=1}^{n} r(t_i)\right]^2$ by substiuting in the probability limit of $\frac{1}{n}\sum_{i=1}^{n} r(t_i)$ (that is equal to zero).

Question B.4: Assume that T is some positive integer ($T \in \mathbb{N}$), and that we have $n = T$ observations of the returns, that is we have a sample $(r(t))_{t=1,\dots T}$ with $r(t) = p(t) - p(t-1)$. Let

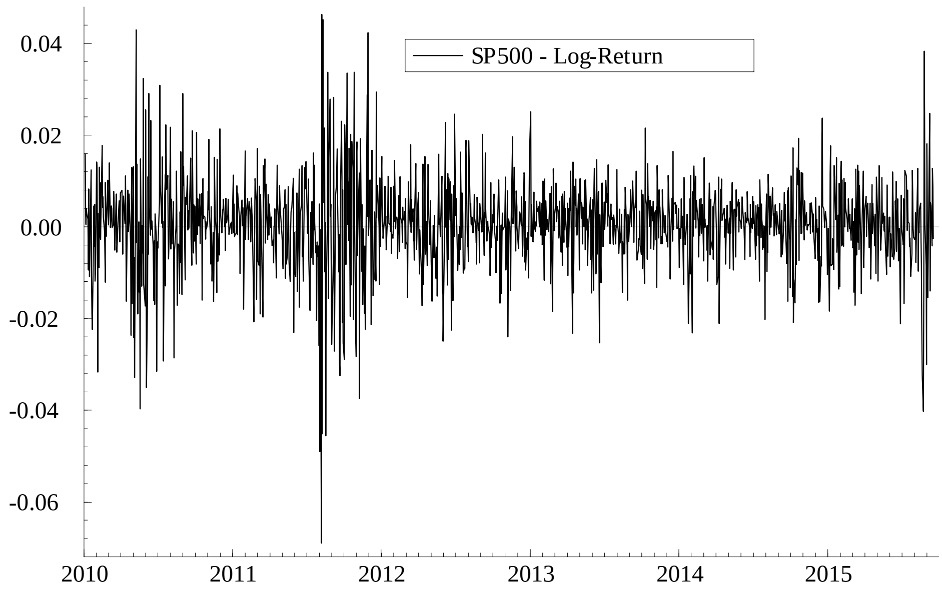
$$\hat{\gamma}_T = \frac{1}{T}\sum_{t=1}^{T} r(t),$$

and argue that as $T \to \infty$,

$$\sqrt{T}\left(\hat{\gamma}_T - \mu\right) \xrightarrow{d} N(0, \sigma^2).$$

Solution: *Using the properties of the BM, we have that $r(t) \sim i.i.d.N(\mu, \sigma^2)$. By a CLT for i.i.d. processes, we have that $\sqrt{T}\left(\hat{\gamma}_T - \mu\right) \xrightarrow{d} N(0, \sigma^2)$*

Question B.5: The following figure shows the daily log-returns of the S&P 500 index for the period January 4, 2010 to September 17, 2015.



Discuss briefly whether the model in (B.1) is a reasonable model for the daily log returns of the S&P 500 index.

Solution: *The model in (B.1) suggests that daily log-returns* $r(t) = p(t) - p(t-1)$, $t = 1, 2, ...,$ *should be given by*

$$\mu + \sigma(W(t) - W(t-1)).$$

By the properties of the Brownian motion, we would have that $r(t) \sim i.i.d.N(\mu, \sigma^2)$. *I.e. the returns would be independent and Gaussian with constant mean and variance. By visual inspection of the series, it appears that the returns are heteroskedastic, and we know from the lectures that the returns are unconditionally heavy-tailed (i.e. non-Gaussian). This suggests that the model is not appropriate for modelling the main features of the daily return series. Ideally, a few derivations should be included.*